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Optimal regularity of the solutions to the classical obstacle problems

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Abstrac:*Este documento é um modelo para a formatação dos textos de trabalhos, minicursos e oficinas submetidos à XI BIENAL DE MATEMÁTICA, que ocorrerá na Universidade Federal de São Carlos, de 29 de julho a 02 de agosto de 2024.*
O resumo deve ter, no máximo, 250 palavras. (Não alterar o estilo e tamanho das fontes deste documento.)

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1. Introdution

Many phenomena in physics, biology, economics, and financial mathematics can be modeled using partial differential equations. Those problems involving PDEs where the solutions are restricted to the boundary conditions of the domain are called boundary problems. On the other hand, there are some stationary problems, where the region which the diffusion process is a priori unknown, are called free boundary problems. Let us consider a membrane (whose boundary remains fixed) at the top of a given body (an obstacle) under the action of contact forces, e.g., tension, friction, air resistance, and gravity. Actually, this archetype of mathematical models that study the position equilibrium of the membrane is often called *obstacle problem* (cf. [2] e [4]). From a variational point of view, if the membrane is over a defined obstacle, such as the graph of a function $\phi \in H^2(\Omega)\cap C^0(\Omega)$, them, the problem reduces to minimize a functional

$$\mathcal{J}(v) = ⨏_{\Omega} |\nabla v|^2 dx - ⨏_{\Omega} f u dx,\ \forall v \in \mathcal{K}.$$

Where $\mathcal{K} = \{u \in H^1_0(\Omega) : u \geq \phi \ \text{ a.e in } \Omega\}$, were $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\phi \leq 0$ on $\partial\Omega$ and $f \in L^2(\Omega)$ a given function.
We must emphasize that the existence of a minimizer is a well-known fact and occurs through the application of the direct method of calculating variations (see [3]).

We will study the classical obstacle problem, showing the existence and uniqueness of a solution through the classical approach, that is, making use of the tools of calculus of variations. In addition, we will study optimal regularity of the solution. For ease, we assume that $f = 0$. To illustrate our approach, note that if the solution to the obstacle problem is continuous; it follows from the Euler-Lagrange equations that

$$\begin{cases} \Delta u = 0 \ \text{ in } \ u \geq 0 \\ \Delta u = \Delta\phi \ \text{ in } u = 0 \\ u \geq \phi \ \text{ in } \Omega \end{cases}$$

Now, due to the linearity of $-\Delta$, we reduce the problem of obstacles to the case of a zero obstacle due to the change of variable

$$\mathbf{u} = u - \phi$$

2. Resultados obtidos

Suppose that $\phi \in C^{1,1}(\Omega)$, therefore $\mathbf{u} \in C^{1,\alpha}(\Omega) \cap H^2(\Omega)$, for every $\alpha < 1$. Let us now consider the open set

$$\mathcal{O} := \{x \in \Omega : \mathbf{u}(x) > 0\},$$

it follows that solution $\mathbf{u} \in H^2(\Omega) \cap C^0(\Omega)$, $\mathbf{u} \geq 0$ in Ω satisfies

$$\Delta \mathbf{u} = -\Delta\phi \chi_{\mathcal{O}} \text{ in } \Omega. \tag{1}$$

Lemma Let $\hat{x} \in \Omega$ and $\eta > 0$ be such that $B_{2\eta}(\hat{x}) \subset \Omega$. If w satisfies $\Delta w = \Delta\phi$ in $B_{2\eta}(\hat{x})$, then

$$\|D^2(w)\|_{L^\infty(B_{2\eta}(\hat{x}))} \leq C\left(\frac{1}{\eta}\|w\|_{L^\infty(B_{2\eta}(\hat{x}))} + \|D^2(\phi)\|_{L^\infty(B_{2\eta}(\hat{x}))}\right),$$

for some $C = C(n) > 0$.

Lemma [**Harnack inequality**] Let $w \in H^2(\Omega)$ satisfies $\Delta w = 0$ in Ω, $w \geq 0$ in Ω. Then, for any $\hat{x} \in \Omega$ e $\eta > 0$ such that $B_{2\eta}(\hat{x}) \subset \Omega$, we have that

$$\sup_{x \in B_\eta(\hat{x})} w(x) \leq C \inf_{x \in B_\eta(\hat{x})} w(x),$$

for some universal constant C = C(n) > 0.

Lemma [**Non-degeneracy of the solution**] Let $\hat{x} \in \partial\mathcal{O} \cap \Omega$ and $\eta > 0$ such that $B_\eta(\hat{x}) \subset \Omega$. Then,

$$\sup_{x \in B_\eta(\hat{x})} \mathbf{u}(x) \geq C\eta^2,$$

for some constant C = C(n) > 0.

Theorem (*Regularity* $C^{1,1}$) Let $\mathbf{u} \in H^2(\Omega) \cap C^0(\Omega)$, $u \geq 0$ in Ω be a solution to (1). Then, $u \in C^{1,1}_{loc}(\Omega)$ and

$$\|D^2(\mathbf{u})\|_{L^\infty(K)} \leq C\left(\frac{1}{\eta}\|\mathbf{u}\|_{L^\infty(\Omega)} + \|D^2(\phi)\|_{L^\infty(\Omega)}\right),$$

for any $K \Subset \Omega$, where $C = C(n, d(K, \partial\Omega))$.
The optimal regularity of the solution to the obstacle problem is obtained through the inequality obtained in the previous theorem.

Sejam $\hat{x} \in \partial\mathcal{O} \cap \Omega$ fixo e $\eta > 0$ tal que $B_{2\eta}(\hat{x}) \subset \Omega$. Consideremos $\mathbf{u} = u_1 + u_2 \in B_{2\eta}(\hat{x})$, onde

$$\begin{aligned} \Delta u_1 = \Delta\mathbf{u},\ \ \Delta u_2 = 0 \ \text{ em } \ B_{2\eta}(\hat{x}) \\ u_1 = 0,\ \ u_2 = \mathbf{u} \ \text{ no } \ \partial B_{2\eta}(\hat{x}) \end{aligned}$$

Seja $\rho := \|\Delta\phi\|_{L^\infty(\Omega)}$. Notemos que

$$|\Delta u_1(x)| \leq \rho, \tag{2}$$

para cada $x \in B_{2\eta}(\hat{x})$. Por outro lado, consideremos a função

$$\psi : x \longmapsto \frac{1}{2n}\left(4\eta^2 - |x - \hat{x}|^2\right).$$

Observemos que $\psi(x) = 0$, para todo $x \in \partial B_{2\eta}(\hat{x})$. Além disso,

$$\begin{cases} -\Delta\psi & = 1 \ \text{ em } \ B_{2\eta}(\hat{x}) \\ \psi & = 0 \text{ em } \partial\ B_{2\eta}(\hat{x}). \end{cases}$$

Logo, por (2) temos que

$$\Delta\rho\psi(x) \leq \Delta u_1(x) \leq -\Delta\rho\psi(x), \forall x \in B_{2\eta}(\hat{x}).$$

Como $\psi(x) = u_1(x) = 0$ para toda $x \in \partial B_{2\eta}(\hat{x})$, pelo Principio de Comparação, tem-se que

$$-\rho\psi(x) \leq u_1(x) \leq \rho\psi(x), \forall x \in B_{2\eta}(\hat{x}).$$

Logo, pela definição de ψ concluímos que

$$|u_1(x)| \leq \rho\psi(x) \leq \frac{2}{n}\eta^2\rho, \forall x \in B_{2\eta}(\hat{x}).$$

Dado que u_2 é harmônica em $B_{2\eta}(\hat{x})$ e $u_2 = \mathbf{u} = 0$ em $\partial B_{2\eta}(\hat{x})$, pelo **??** (Princípio de Comparação) $u_2 \geq 0$ em $B_{2\eta}(\hat{x})$. Dado que $\hat{x} \in \partial\mathcal{O} \cap \Omega$, temos que $\mathbf{u}(\hat{x}) = 0$, logo por (2),

$$u_2(\hat{x}) = -u_1(\hat{x}) = 0 \leq \frac{1}{n}\eta^2\rho.$$

Agora, aplicando a Desigualdade de Harnack, segue que

$$\sup_{x \in B_\eta(\hat{x})} u_2(x) \leq C \inf_{x \in B_\eta(\hat{x})} u_2(x) \leq C u_2(\hat{x}) \leq \frac{\mathrm{C}}{n}\eta^2\rho,$$

para alguma constante C = C(n) > 0 O resultado anterior é obtido de forma análoga para u_1 e como $\mathbf{u}$ é uma combinação linear de u_1 e u_2, temos o resultado desejado.

Referências

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