An algorithm for solving the problem $\max\{cx : x^t Q_n(a) x \leq 1, x \in \mathbb{Z}_+^n\}$

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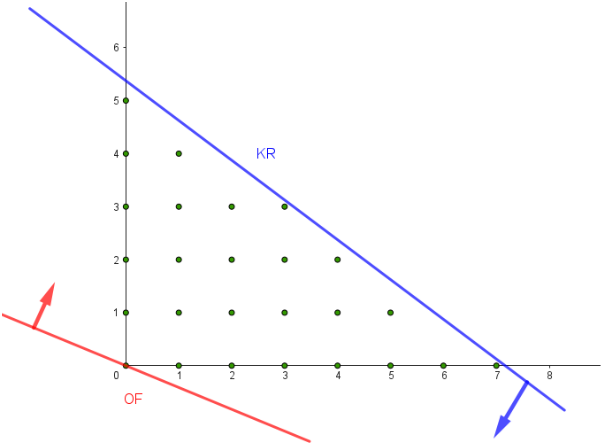
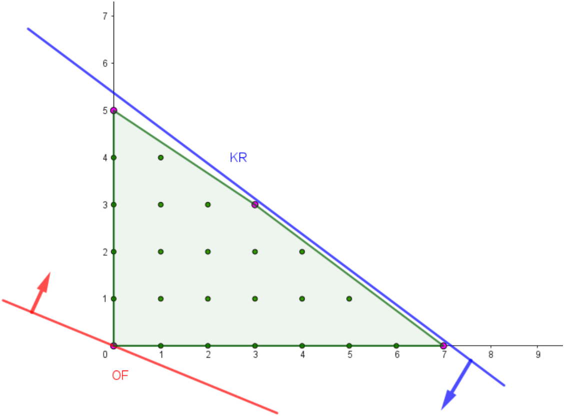
Content

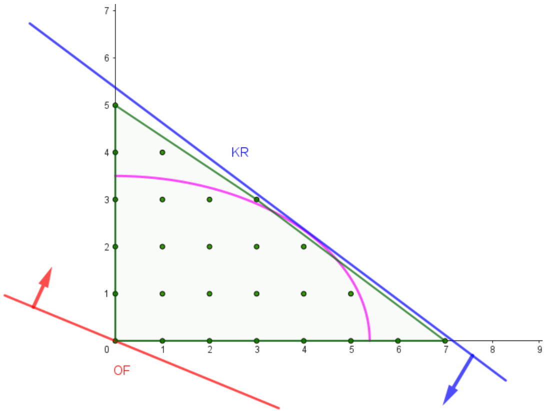
The Knapsack Problem



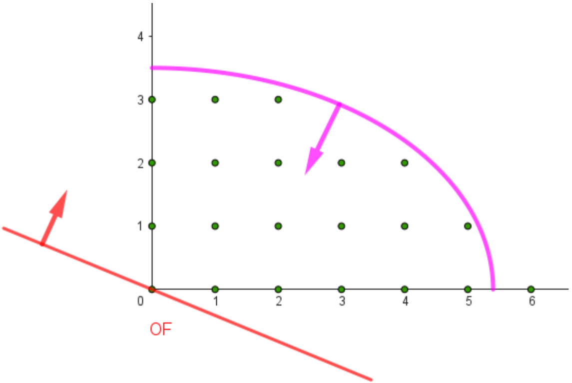
Convex Hull



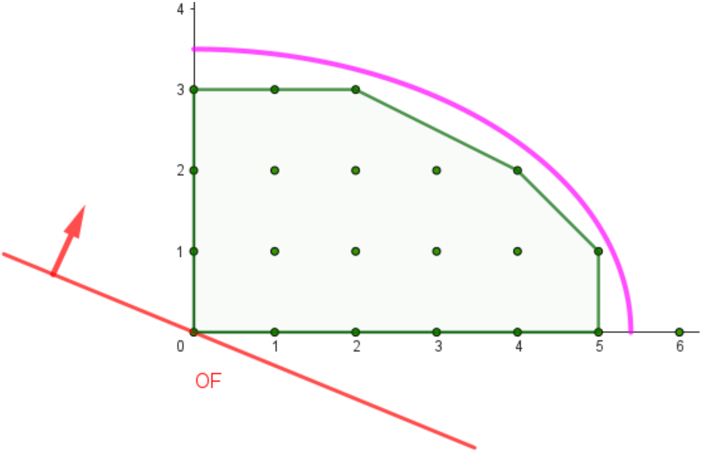
The idea



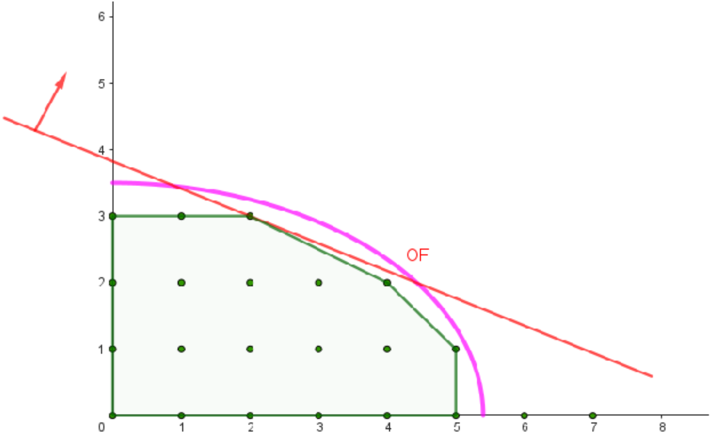
The idea



The idea



The idea



Notation

In this work we use the following notations: given any $n \in \mathbb{N}$,

* $\mathbb{R}^n_{++}$ denotes the vectors of $\mathbb{R}^n$ with positive components;
* For $r \in \mathbb{R}$, $\lfloor r \rfloor$ denotes the integer part of r
* $\hat{0} \in \mathbb{R}^n$ and $\hat{1} \in \mathbb{R}^n$ the vectors with all components equal to 0 and 1, respectively;
* $\{e_i\}_{j=1}^n$ denotes the canonical base of $\mathbb{R}^n$;
* For $a \in \mathbb{R}^n_{++}$, we denote by $Q_n(a)$ the matrix defined by $Q_n(a)_{i,i} = \frac{1}{a_i^2}$ and $Q_n(a)_{i,j} = 0$ if $i \neq j$, for all $i, j \in \{1, \ldots, n\}$.
* $E(Q_n(a)) = \{x \in \mathbb{R}^n_+ : xQ_n(a)x^t \leq 1\}$.

$IOE(Q_n(a))$

Optimização Problem

Let be $a, c \in \mathbb{R}^n_{++}$*, we consider the problem* $IOE(Qn(a))$ *defined as*

$$\max \sum_{i=1}^{n} c_i x_i$$

$$s.t: \ \sum_{i=1}^{n} \frac{x_i^2}{a_i} \leq 1$$

$$x_i \in \mathbb{Z}_+, i \in \{1, \dots, n\}$$

State of art

WITZGAL in 1963 [3]

Linear objective to linear and parabolic constraints

$$\min \sum_{i=1}^{n} c_i x_i - c_0$$

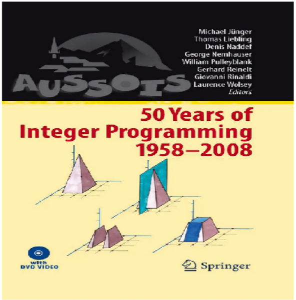
$$s.t: \; P_j(x_1, \ldots, x_n) \geq 0, \; j \in \{1, \ldots, m\}$$

$$x_i \in \mathbb{Z}_+, i \in \{1, \ldots, n\}$$

where

$$P_j(x_1, \ldots, x_n) = a_{j00} - L_{j0} - \sum_{s=1}^{k} b_{js}(L_{js})^2(x_1, \ldots, x_n);$$
$$L_{js}(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_{jsi} x_i, s \in \{0, \ldots, k\}, j \in \{1, \ldots, m\}$$

State of art



BALINSKI:Integer Programming: Methods, Uses, Computation.

State of art

MADRIZ E.; SICRE, M.; ROCHA, C, MACULAN, N.:A in 2024 [2]

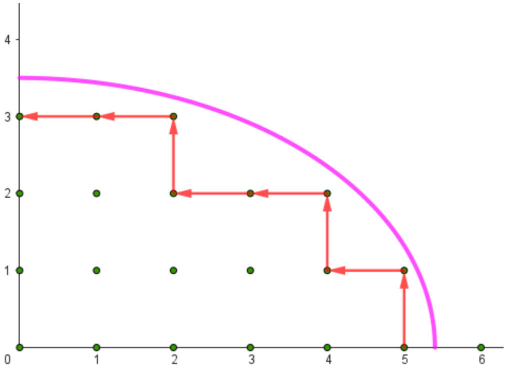
Pseudopolynomial algorithm

$$\max \sum_{i=1}^{n} c_i x_i$$

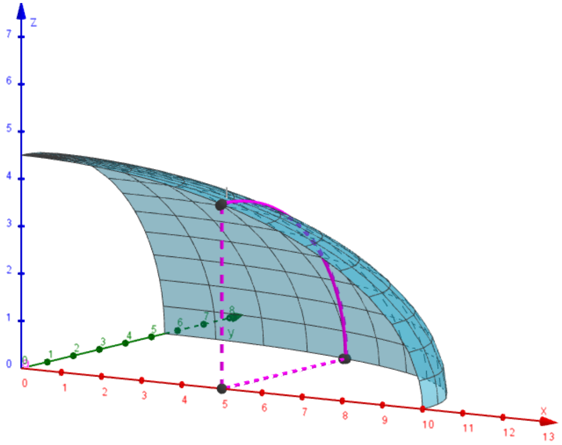
$$s.t: \ \sum_{i=1}^{n} x_i^2 \leq 1$$

$$x_i \in \mathbb{Z}_+, i \in \{1, \ldots, n\}$$

The heart of the algorithm



The algorithm in $\mathbb{R}^3$



The functions for the algorithm

For any $m \in \mathbb{N}$ with $m \geq 2$, and given $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_m) \in \mathbb{R}^m_{++}$, we will consider the functions $f_{\bar{a}} : E(Q_{m-1})(\bar{a}) \to \mathbb{R}$ and $g_{\bar{a}} : E(Q_m(\bar{a})) \to \mathbb{R}$,

$$f_{\bar{a}}(x_1, \ldots, x_{m-1}) = \left\lfloor \bar{a}_m \left(1 - \sum_{i=1}^{m-1} \frac{x_i^2}{\bar{a}_i^2}\right)^{\frac{1}{2}} \right\rfloor \tag{1}$$

$$g_{\bar{a}}(x_1, \ldots, x_m) = \left\lfloor \bar{a}_m \left(1 - \sum_{i=1}^{m-1} \frac{x_i^2}{\bar{a}_i^2}\right)^{\frac{1}{2}} - x_m \right\rfloor . \tag{2}$$

The functions for the algorithm

Observation

Note that, for a given $x \in E(Q_{m-1}(\bar{a}))$ *and* $y \in E(Q_m(\bar{a}))$

1 $f_{\bar{a}}(x)$ *gives the largest integer such that*

$$(x, f_{\bar{a}}(x)) \in E(Q_{m-1}(\bar{a}));$$

2. $g_{\bar{a}}(y)$ *gives the largest integer such that*

$$y + g_{\bar{a}}(y)e_m \in E(Q_m(\bar{a})).$$

The $AIOE(Q_n(a))$

Algorithm 1 $AIOE(Q_n(a))$

Data: $n \in \mathbb{N}$, $a \in \mathbb{R}^n_{++}$, $c \in \mathbb{R}^n_{++}$
Result: p^* the optimal solution of the $IOE(Q_n(a))$ problem
Initialize: $p^* = \hat{0} \in \mathbb{R}^n$
if $n = 2$ **then**
  $w^0 = (\lfloor a_1 \rfloor + 1, 0)$
  for $k = 1$ to $(\lfloor a_1 \rfloor + 1)$ **do**
    $w^k = w^{k-1} - e_1 + g_a(w^{k-1} - e_1)e_2$
    $p^* = argmax\{cp^*, cw^k\}$
  end for
else
  for $i_1 = 0$ to $\lfloor a_1 \rfloor$ **do**
    for $i_2 = 0$ to $f_{(a_1,a_2)}(i_1)$ **do**
      $\vdots$
      for $i_{n-3} = 0$ to $f_{(a_1,\dots,a_{n-3})}(i_1, \dots, i_{n-4})$ **do**
        for $i_{n-2} = 0$ to $f_{(a_1,\dots,a_{n-2})}(i_1, i_2, \dots, i_{n-3})$ **do**
          $s = f_{(a_1,\dots,a_{n-1})}(i_1, i_2, \dots, i_{n-2}) + 1$
          $w^0 = (i_1, \dots, i_{n-2}, s, 0)$
          for $k = 1$ to s **do**
            $w^k = w^{k-1} - e_{n-1} + g_a(w^{k-1} - e_{n-1})e_n$
            $p^* = argmax\{cp^*, cw^k\}$
          end for
        end for
      end for
      $\vdots$
    end for
  end for
end if

The $AIOE(Q_n(a))$

```
Initialize: p* = 0̂ ∈ ℝ^n
if n = 2 then
    w^0 = (⌊a_1⌋ + 1, 0)
    for k = 1 to (⌊a_1⌋ + 1) do
        w^k = w^{k-1} − e_1 + g_a(w^{k-1} − e_1) e_2
        p* = argmax{cp*, cw^k}
    end for
```

$$
\begin{array}{l}
\textbf{Initialize: } p^* = \hat{0} \in \mathbb{R}^n \\
\textbf{if } n = 2 \textbf{ then} \\
\quad w^0 = (\lfloor a_1 \rfloor + 1, 0) \\
\quad \textbf{for } k = 1 \textbf{ to } (\lfloor a_1 \rfloor + 1) \textbf{ do} \\
\qquad w^k = w^{k-1} - e_1 + g_a(w^{k-1} - e_1)e_2 \\
\qquad p^* = argmax\{cp^*, cw^k\} \\
\quad \textbf{end for}
\end{array}
$$

The $AIOE(Q_n(a))$

else
  for $i_1 = 0$ to $\lfloor a_1 \rfloor$ **do**
    for $i_2 = 0$ to $f_{(a_1,a_2)}(i_1)$ **do**
      $\vdots$
      for $i_{n-3} = 0$ to $f_{(a_1,\ldots,a_{n-3})}(i_1,\ldots,i_{n-4})$ **do**
        for $i_{n-2} = 0$ to $f_{(a_1,\ldots,a_{n-2})}(i_1,i_2,\ldots,i_{n-3})$ **do**
          $s = f_{(a_1,\ldots,a_{n-1})}(i_1,i_2,\ldots,i_{n-2}) + 1$
          $w^0 = (i_1,\ldots,i_{n-2},s,0)$
          for $k = 1$ to s **do**
            $w^k = w^{k-1} - e_{n-1} + g_a(w^{k-1} - e_{n-1})e_n$
            $p^* = argmax\{cp^*, cw^k\}$
          end for
        end for
      end for
      $\vdots$
    end for
  end for
end if

Convergence

Theorem

The algorithm $AIOE(Q_n(a))$ finds an optimal solution of the $IOE(Q_n(a))$ problem

Convergence

Proof

We will consider the case when $n = 2$.

1. *Let $w = (w_1, w_2)$ be an optimal solution of the $IOE(Q_2(a))$;*
2. *Notice that it hold $w_1 \leq \lfloor a_1 \rfloor + 1$ Hence, for some $w_1 \leq \lfloor a_1 \rfloor + 1$;*
3. *For some $k \in \mathbb{N}$ with $k \leq \lfloor a_1 \rfloor + 1$, it holds $w_1^k = w_1$;*
4. *Since w is optimal for the $IOE(Q_2(a))$ problem and $c \in \mathbb{R}^2_{++}$, it follows that $w_2 \geq w_2^k$.*
5. *From the definition of the function g_a in (2) with $\bar{a} = a$, it follows that $w_2 \leq w_2^k$, theefore $w^k = w$.*

Convergence

Proof

We will consider the general case.

1. *Let* $w =$ *be an optimal solution of the* $IOE(Q_n(a))$
2. *From the definition of the functions* $f_{\bar{a}}$ *in* (1) *with* $\bar{a} = \bar{a}^k$ *as in the algorithm, and the fact that* $w \in E(Q_n(a)) \cap \mathbb{Z}_+^n$*, it follows that* $w_1 \leq \lfloor a_1 \rfloor$ *and* $w_k \leq f_{(a_1,\ldots,a_k)}(w_1, \ldots, w_{k-1})$ *for* $k = 1, \ldots, n-2$.
3. *For* $k = 1, \ldots, n-2$, *the index* i_k *will take the value* w_k.
4. *Notice that only the last two components of the vectors* w^k *change along these iterations.*

Convergence

Proof

5. *Simple calculations show that in these two components we have iterations of the algorithm* $AIOE(Q_n(a))$, *applied to the resolution of the problem* $\max\{(c_{n-1}, c_{n-2})x : x \in E(Q_n(\bar{a})) \cap \mathbb{Z}_+^2\}$, *with*

$$\bar{a} = \left(1 - \sum_{k=1}^{n-2} (i_k^2 / a_k^2)\right)^{\frac{1}{2}} (a_{n-1}, a_n)$$

.

6. *We will obtain the* (w_{n-1}, w_n), *which is optimal.* □

Muito obrigado !!!!

Bliography

- Balinski, M. . Integer Programming: Methods, Uses, Computation. In: Jünger, M., et al. 50 Years of Integer Programming 1958-2008. Springer, Berlin, Heidelberg. (2010)
- MADRIZ E.; SICRE, M.; ROCHA, C, MACULAN, N.:A pseudopolynomial algorithm for solving the problem $\max\{cx : ||x|| \leq r, x \in \mathbb{Z}_+^n\}$, submitted (2024)
- WITZGALL, C.: An all-integer programming algorithm with parabolic constraints. Soc. Ind. and Appli. Math. 11, 855?870 (1963)